21 Sorting presents

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Project: GV-AP16 – *Computational and structural aspects of point set surfaces*
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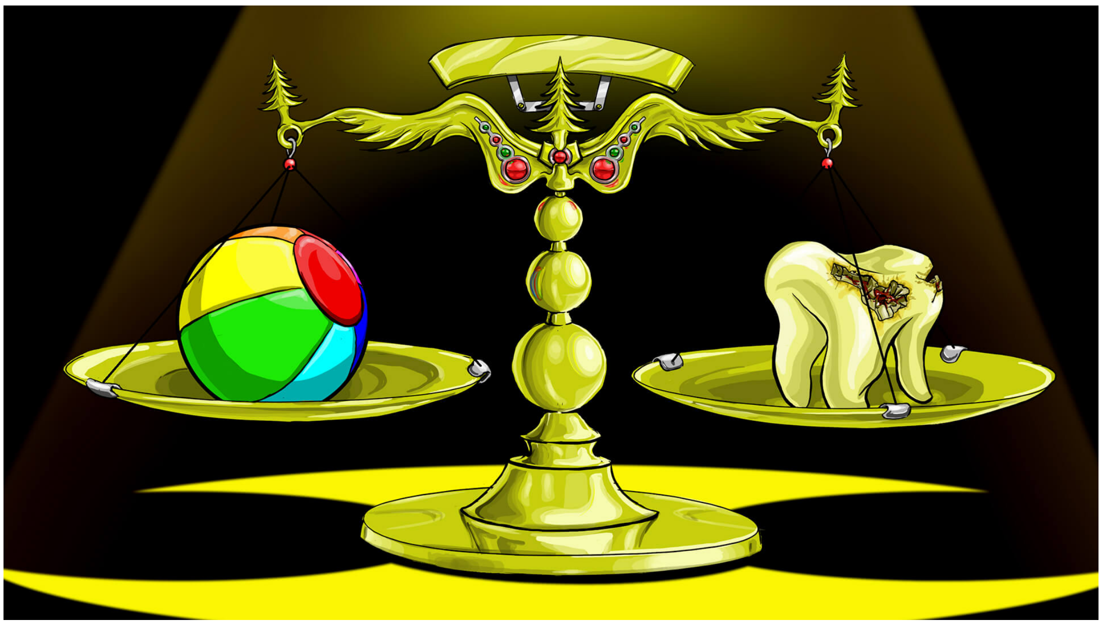
21.1 Challenge

Until now, the gnomes of the village used to sort the packages of Christmas presents by the number of toys inside. After several complaints filed by the World Dentist Association, the gnomes have to consider the amount of sweets in each package too. Thus, every package P gets a toy rating T_P and a sugar rating S_P. These ratings are distinct for any two packages. Now, the gnomes want to stack the packages according to the following rules:

- For two packages P and Q placed next to each other, one has: P lies to the left of Q if and only if $T_P < T_Q$.
- For two packages P and Q placed one top of each other, one has: P lies beneath Q if and only if $S_P < S_Q$.

A first delivery of nine cubic packages arrives; they are to be sorted in three rows, each consisting of three packages. The packages for the children are assembled such that less toys are compensated for with more sweets. The ratings (T_P,S_P) are the following: (1,9),(2,8),(3,7),...,(9,1).

How many possibilities are there to stack these nine packages as a stack of size 3×3 according to the two stack rules above? (Note: It is out of question whether one can regard the stacks from both sides, because *left* is uniquely defined only from one side.)



Possible answers:

1. 1
2. 2
3. 3
4. 9
5. 11
6. 12
7. 13
8. 16

9. 21
10. 42

Project relevance:

This question arises when considering the combinatoric order of geometric point clouds. If one wants to order the n^2 points $(x_1,y_1),\ldots,(x_{n^2},x_{n^2})$, $n \in \mathbb{N}$, in a grid of size $n \times n$ such that the x-coordinate increases in the rows from left to right and the the y-coordinate increases in the columns from the bottom up, there exist only finitely many combinatoric possibilities to fill the grid—independent of the specific coordinates of the points. The grid is applied to determine neighbourhoods. This is of importance, for instance, for simulations of biological cells and physical particles or when processing large amounts of data provided by 3D scanners.

21.2 Solution

The correct answer is: 10.

To identify the correct answer, we rephrase the problem first: Assume we have found an order of the packages obeying both rules (see Figure 11, left). Now, we change the rating of each package from (T,S) to ($T, 10 - S$) (see Figure 11, center). Then, each package is now rated (T,T). However, the order of the packages is not correct anymore and we have to interchange the row at the bottom with the row at the top (see Figure 11, right). With this transformation, any order of the given ratings (1,9),(2,8), . . . ,(9,1) may be converted into an order of of the ratings (1,1), (2,2), . . . , (9,9)—and vice versa. Thus, there is a bijection between these two sets of possible orders, and these sets need to be of the same size. If we know how many orders there exist for the ratings of the form (T,T), we also know this quantity for our original ratings (T,$10 - T$).

(1,9)	(2,8)	(7,3)
(3,7)	(5,5)	(8,2)
(4,6)	(6,4)	(9,1)

(1,1)	(2,2)	(7,7)
(3,3)	(5,5)	(8,8)
(4,4)	(6,6)	(9,9)

(4,4)	(6,6)	(9,9)
(3,3)	(5,5)	(8,8)
(1,1)	(2,2)	(7,7)

Figure 11: Left: order of the packages with original ratings. Center: change of rating. Right: correct order according to new rating.

Now, we know that $T_P = S_P$ for all packages P. Furthermore, $T_P \neq T_Q$ for any two packages P and Q. Since we are only concerned with the order, we can set $T_{P_1} = 1$, $T_{P_2} = 2$, ..., $T_{P_9} = 9$. The problem reduces to the question how the integers 1 to 9 can be written in a grid of size 3×3 such that the rows from left to right and the columns from bottom to top are ordered increasingly. Here, the total number of 9 packages is partitioned into (3,3,3).

More generally, one can consider, for a given positive integer N, a partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ with $\lambda_1 + \ldots + \lambda_n = N$ and $\lambda_i > \lambda_j$ whenever $i > j$. The grid with λ_i cells in the i-th row is called *Ferrers diagram* (see Figure 12, left). The number of possibilities to fill up this grid with the numbers from 1 to N, such that every row from left to right and every column from bottom to top is ordered increasingly, is provided by the *hook-length formula*:

$$\frac{N!}{\prod h_\lambda(i,j)},$$

where, for each cell (i,j) in the grid, the *hook* $H_\lambda(i,j)$ is the set of cells (a,b) such that $a = i$ and $b \geq j$ or $a \geq i$ and $b = j$. The *hook-length* $h_\lambda(i,j)$ is the number of cells in the hook $H_\lambda(i,j)$ (see Figure 12, right).

In the special case $N = 9$ and $\lambda = (3,3,3)$, we have

$$h_\lambda(i,j) = (3 - i) + (3 - j) + 1,$$

and the number of possible orders is

$$\frac{9!}{\prod_{i=1}^{3}\prod_{j=1}^{3}\left((3-i)+(3-j)+1\right)} = 42.$$

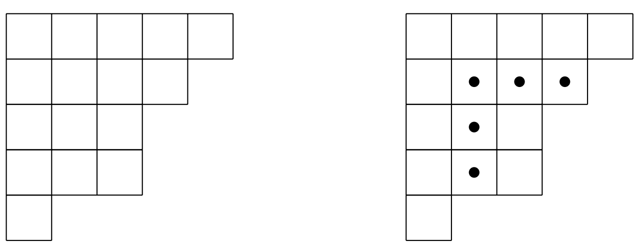


Figure 12: Left: Ferrers diagram for the partition $\lambda = (5,4,3,3,1)$ of $N = 16$. Right: according hook $H_\lambda(2,2)$ with hook-length $h_\lambda(2,2) = 5$.

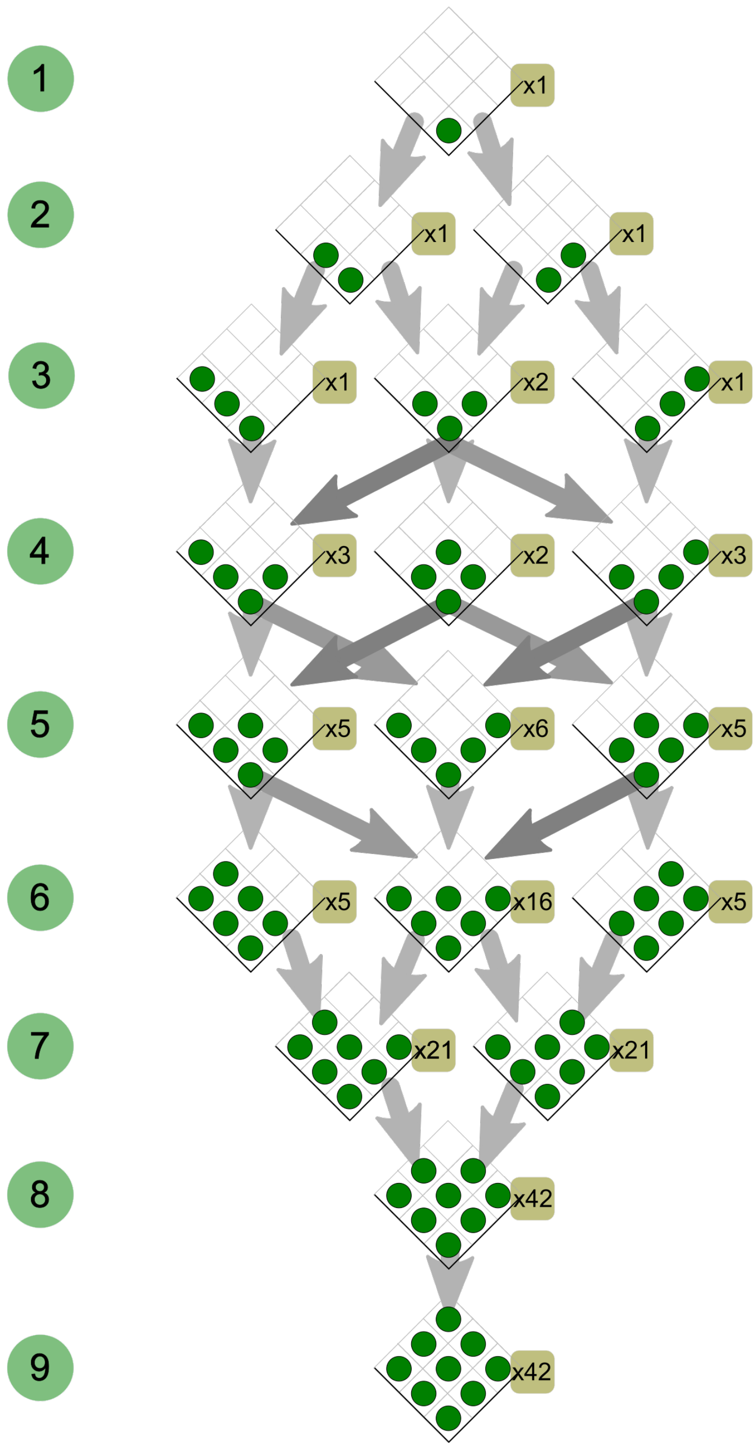
Further remarks: For a quadratic grid of side length 1, 2 or 3 it is known, for which specific ratings there are the least or the most possible orders, respectively. For $n \in \{1,2,3\}$, there are ratings that allow a unique order. This is not true for $n \geq 4$. It is conjectured that the rating $(1,1),\ldots(n^2,n^2)$ is the one which allows the most orders. This fact is known for $n \in \{1,2,3\}$, but remains unproven for $\neq 4$.

Alternative solution:

Those who are not acquainted with the hook-length formula are also able to compute the number of possiblities: We assume that the ratings of the packages have been already converted to (1,1), . . . , (9,9). Thus, we can represent each package by an integer from 1 to 9.

For a better understanding, we rotate the quadratic grid by 45° such that it stands on the bottom left vertex. We know let the packages fall—as under gravitational force—into the grid of size 3×3. The only possible position for package 1 to come to rest at is the bottom of the grid, because other packages are not allowed to be placed underneath the package with rating 1. Afterwards, we gradually drop the packages 2 to 9. These may settle at different positions in the grid. However, these positions are restricted by the edges of the grid and the packages that have already fallen into the grid.

In the following diagram, the possible valid configurations are displayed; the arrows indicate how one gets from one valid configuration to another. At times there are various ways to get to the same configuration. Therefore, the number of ways to get to a configuration is displayed besides the configuration itself. At the bottom, you can read off that there are 42 legit ways to get to a full configuration of our quadratic grid of size 3×3.


1
x1
2
x1
x1
3
x1
x2
x1
4
x3
x2
x3
5
x5
x6
x5
6
x5
x16
x5
7
x21
x21
8
x42
9
x42